HARMONIC MANIFOLDS AND THE VOLUME OF TUBES ABOUT CURVES

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Abstract. H. Hotelling proved that in the n-dimensional Euclidean or spherical space, the volume of a tube of small radius about a curve depends only on the length of the curve and the radius. A. Gray and L. Vanhecke extended Hotelling's theorem to rank one symmetric spaces computing the volumes of the tubes explicitly in these spaces. In the present paper, we generalize these results by showing that every harmonic manifold has the above tube property. We compute the volume of tubes in the Damek–Ricci spaces. We show that if a Riemannian manifold has the tube property, then it is a 2-stein D'Atri space. We also prove that a symmetric space has the tube property if and only if it is harmonic. Our results answer some questions posed by L. Vanhecke, T. J. Willmore, and G. Thorbergsson.

1. Introduction

In 1939 H. Hotelling [1] showed that in the n-dimensional Euclidean or spherical space, the volume of a tube of small radius about a curve depends only on the length of the curve and the radius. Hotelling's result was generalized in different directions. H. Weyl [2] proved that the volume of a tube of small radius about a submanifold of a Euclidean or spherical space depends only on intrinsic invariants of the submanifold and the radius. A. Gray and L. Vanhecke [3] extended Hotelling's theorem to rank one symmetric spaces.

In the present paper, we are interested in the widest class of connected Riemannian manifolds to which Hotelling's theorem can be extended. The problem of finding this class was raised during private discussions of the first author with G. Thorbergsson at the University of Cologne in 2013. The question is motivated by a new characterization of harmonic manifolds given by the authors [4]. Harmonic manifolds were introduced by E. T. Copson and H. S. Ruse [5] as Riemannian manifolds having a non-constant radially symmetric harmonic function in the neighborhood of any point. They proved that this condition holds if and only if small geodesic spheres have constant mean curvature. By the results of the paper [4], connected harmonic manifolds are characterized also by the property that the volume of the intersection of two geodesic balls of small equal radius depends only on the radius and the distance between the centers of the balls. As we shall see in Section 3, this characterization implies that every connected harmonic manifold has Hotelling's tube property. It seems to be a reasonable conjecture that the converse is also true, that is, Hotelling's tube property is true only in harmonic manifolds. A weaker form of this conjecture, saying that a symmetric space has the tube property if and only if it is harmonic, was proposed by G. Thorbergsson.

The structure and the main results of the paper are the following. In Section 2, we compute a formula for the volume of tubes about a curve in a general Riemannian manifold. This formula is not new, an equivalent formula appears also in [6]. The formula gives the volume as an integral with respect to the arc length parameter of the curve, where the integrand is the sum of two terms, one of which depends only on the velocity vector of the curve, while the other depends on both the velocity and the acceleration vectors. L. Vanhecke and T. J. Willmore [6] proved that the second term vanishes in D'Atri spaces and conjectured that if the second term vanishes for all curves, then the space must be a D'Atri space. Recall that a Riemannian manifold is called a D'Atri space if its local geodesic symmetries are volume preserving. At the end of Section 2, we prove this conjecture, and as a consequence, we obtain that a Riemannian manifold has the tube property if and only if it is a D'Atri space and satisfies the tube property for tubes about geodesic curves (Theorem 1).

The main result of Section 3 is Theorem 2, claiming that every connected harmonic manifold has the tube property.

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There are two known classes of harmonic manifolds: two-point homogeneous spaces and Damek–Ricci spaces. These examples exhaust all *homogeneous* harmonic manifolds according to J. Heber [7]. Two-point homogeneous spaces are the Euclidean and the rank one symmetric spaces. For these spaces, the volume of tubes about curves were computed explicitly in [3]. We complete the picture by computing the volume of tubes about curves in the Damek–Ricci spaces in Section 4.

In Sections 5 and 6, we prove some facts supporting the conjecture that the tube property can hold only in harmonic manifolds. Harmonic manifolds are real analytic Riemannian manifolds, and they can be characterized among real analytic Riemannian manifolds by a sequence $\{L_k \mid k \geq 2\}$ of curvature conditions, known as Ledger conditions, see [8] and [9, section 6.8]. It would be enough to show that the tube property implies even Ledger conditions $\{L_{2k} \mid k \geq 1\}$ for two independent reasons. First, L. Vanhecke [10] proved that the odd Ledger conditions follow from the even ones. The second reason is that D'Atri spaces satisfy all Ledger conditions of odd order. Conditions L_2 and L_4 are equivalent to the requirement that the manifold is 2-stein. The main result of Section 5 is Theorem 4, claiming that every connected Riemannian manifold having the tube property is 2-stein. In particular, such a manifold is Einstein.

In Section 6, first we adapt the formula for the volume of tubes about curves for the special case of geodesic curves in a symmetric space. Slightly different, but equivalent forms of some of our formulae were obtained by X. Gual-Arnau and A. M. Naveira [11], [12]. In the second part of the section, we verify the above conjecture within the class of symmetric spaces (Theorem 5). 2-stein symmetric spaces were classified by P. Carpenter, A. Gray, and T. J. Willmore [13]. According to the classification, besides the harmonic symmetric spaces, the family of 2-stein symmetric spaces contains 23 dual pairs of irreducible symmetric spaces. Though it would be possible to show the failure of the tube property for each non-harmonic example in the list one by one, we shall present a shorter argument that rules out all of them together. The theorem extends obviously to locally symmetric spaces.

Throughout the paper, every manifold is assumed to be connected and of class $\mathcal{C}^\infty$. By the Kazdan–DeTurck theorem [14], the geodesic normal coordinate systems on an Einstein space provide a real analytic atlas, with respect to which the Riemannian metric is real analytic. In fact, the same is true for Riemannian manifolds satisfying the third Ledger condition, in particular for D'Atri spaces, see [15]. Consequently, Riemannian manifolds having the tube property have a natural real analytic structure. This implies, for example, that the volume of a tube about a geodesic in such a space is a real analytic function of the radius.

2. Volume of tubes and D'Atri spaces

Let (M, g) be a Riemannian manifold. Denote by $\mathrm{Exp}\colon TM \to M$ its exponential map, and by $\mathrm{Exp}_p\colon T_pM \to M$ the restriction of Exp to the tangent space at $p \in M$. Let ∇ be the Levi-Civita connection of M. For a vector field X along a curve γ, we shall use the notation X' for the covariant derivative $\nabla_{\gamma'}X$.

Definition. For a smooth injective regular curve $\gamma\colon [a,b] \to M$ and $r > 0$, set

$$T(\gamma, r) = \{\mathbf{v} \in TM \mid \exists t \in [a,b] \text{ such that } \mathbf{v} \in T_{\gamma(t)}M, \mathbf{v} \perp \gamma'(t), \text{ and } \|\mathbf{v}\| \leq r\}.$$

Assume that r is small enough to guarantee that the exponential map is defined and injective on $T(\gamma, r)$. Then we define the tube of radius r about γ by

$$\mathcal{T}(\gamma, r) = \mathrm{Exp}(T(\gamma, r)).$$

Definition. We say that a Riemannian manifold has the tube property if there is a function $V\colon [0,\infty) \to \mathbb{R}$ such that

$$\mathrm{vol}(\mathcal{T}(\gamma, r)) = V(r)l_\gamma$$

for any smooth injective regular curve γ of length l_γ and any sufficiently small r.

Consider a smooth injective unit speed curve $\gamma\colon [0,l] \to M$ and an orthonormal frame $E_1, \ldots, E_n$ along γ such that $E_n(t) = \gamma'(t)$. Denote by B_r^{n-1} the closed ball of radius r about the origin in $\mathbb{R}^{n-1}$ and by S_r^{n-2} its boundary sphere. Parameterize the tube $\mathcal{T}(\gamma, r)$ by the map $\mathbf{r}\colon B_r^{n-1} \times [0,l] \to M$ defined by the formula

$$\mathbf{r}(x_1, x_2, \ldots, x_n) = \mathrm{Exp}_{\gamma(x_n)}(x_1E_1(x_n) + \cdots + x_{n-1}E_{n-1}(x_n)). \tag{1}$$

The volume of the tube is

$$\operatorname{vol}(\mathcal{T}(\gamma,r)) = \int_{B_r^{n-1}\times[0,l]} \|\partial_1\mathbf{r}\wedge\cdots\wedge\partial_n\mathbf{r}\|(\mathbf{x})\,\mathrm{d}\mathbf{x}. \tag{2}$$

To calculate the partial derivatives of the map $\mathbf{r}$ at a point $\mathbf{x}=(x_1,\dots,x_n)\in B_r^{n-1}\times[0,l]$, consider the Jacobi fields $J_1^{\mathbf{x}},\dots,J_n^{\mathbf{x}}$ along the geodesic $\eta^{\mathbf{x}}\colon[0,1]\to M$, $\eta^{\mathbf{x}}(s)=\mathbf{r}(sx_1,\dots,sx_{n-1},x_n)$, such that $J_i^{\mathbf{x}}(0)=\mathbf{0}$ and $J_i^{\mathbf{x}\prime}(0)=E_i(x_n)$. As these Jacobi fields give the differential of the exponential map, we have $\partial_i\mathbf{r}(\mathbf{x})=J_i^{\mathbf{x}}(1)$ for $i=1,\dots,n-1$.

For the nth partial derivative, consider the geodesic variation $\Gamma(t,s)=\mathbf{r}(sx_1,\dots,sx_{n-1},x_n+t)$. The vector field $J^{\mathbf{x}}(s)=\partial_1\Gamma(0,s)$ is a Jacobi field along the geodesic $\eta^{\mathbf{x}}$, such that $J^{\mathbf{x}}(0)=\gamma'(x_n)$ and $J^{\mathbf{x}\prime}(0)=x_1E_1'(x_n)+\cdots+x_{n-1}E_{n-1}'(x_n)$. This Jacobi field gives the partial derivative of the map $\mathbf{r}$ with respect to its nth variable by $\partial_n\mathbf{r}(x_1,\dots,x_n)=J^{\mathbf{x}}(1)$.

Decompose the Jacobi field $J^{\mathbf{x}}$ into the sum of the Jacobi fields $\hat{J}^{\mathbf{x}},\check{J}^{\mathbf{x}}$ given by the initial conditions $\hat{J}^{\mathbf{x}}(0)=\gamma'(x_n)$, $\hat{J}^{\mathbf{x}\prime}(0)=\mathbf{0}$ and $\check{J}^{\mathbf{x}}(0)=\mathbf{0}$, $\check{J}^{\mathbf{x}\prime}(0)=x_1E_1'(x_n)+\cdots+x_{n-1}E_{n-1}'(x_n)$. We can decompose the Jacobi field $\check{J}^{\mathbf{x}}$ as

$$\check{J}^{\mathbf{x}}=\sum_{i=1}^{n}\left(\sum_{j=1}^{n-1}x_jg(E_j'(x_n),E_i(x_n))\right)J_i^{\mathbf{x}}.$$

Write $\mathbf{x}$ in the form $\mathbf{x}=(\rho y_1,\dots,\rho y_{n-1},y_n)$, where $\|(y_1,\dots,y_{n-1})\|=1$, $\rho\ge 0$, and set $\mathbf{y}=(y_1,\dots,y_n)$. Then we have

$$\begin{aligned}
&\eta^{\mathbf{x}}(s)=\eta^{\mathbf{y}}(\rho s), && \rho J_i^{\mathbf{x}}(s)=J_i^{\mathbf{y}}(\rho s)\text{ for }1\le i\le n,\\
&J^{\mathbf{x}}(s)=J^{\mathbf{y}}(\rho s), && \hat{J}^{\mathbf{x}}(s)=\hat{J}^{\mathbf{y}}(\rho s), \qquad \check{J}^{\mathbf{x}}(s)=\check{J}^{\mathbf{y}}(\rho s).
\end{aligned}$$

The volume density function appearing in (2) can be expressed as

$$\begin{aligned}
\|\partial_1\mathbf{r}\wedge\cdots\wedge\partial_n\mathbf{r}\|(\mathbf{x}) &= \|J_1^{\mathbf{x}}\wedge\cdots\wedge J_{n-1}^{\mathbf{x}}\wedge J^{\mathbf{x}}\|(1)\\
&= \|J_1^{\mathbf{x}}\wedge\cdots\wedge J_{n-1}^{\mathbf{x}}\wedge\hat{J}^{\mathbf{x}}+J_1^{\mathbf{x}}\wedge\cdots\wedge J_{n-1}^{\mathbf{x}}\wedge\check{J}^{\mathbf{x}}\|(1)\\
&= \begin{cases}\rho^{1-n}\|J_1^{\mathbf{y}}\wedge\cdots\wedge J_{n-1}^{\mathbf{y}}\wedge\hat{J}^{\mathbf{y}}+J_1^{\mathbf{y}}\wedge\cdots\wedge J_{n-1}^{\mathbf{y}}\wedge\check{J}^{\mathbf{y}}\|(\rho) & \text{if }\rho>0,\\ 1 & \text{if }\rho=0.\end{cases}
\end{aligned}$$

Extend $E_i(x_n)$ to a parallel vector field $\mathbf{e}_i^{\mathbf{y}}$ along $\eta^{\mathbf{y}}$. If ρ is small, then

$$(J_1^{\mathbf{y}}\wedge\cdots\wedge J_{n-1}^{\mathbf{y}}\wedge\hat{J}^{\mathbf{y}})(\rho)=\rho^{n-1}(\mathbf{e}_1^{\mathbf{y}}\wedge\cdots\wedge\mathbf{e}_n^{\mathbf{y}})(\rho)+O(\rho^{n+1}),$$

and

$$J_1^{\mathbf{y}}\wedge\cdots\wedge J_{n-1}^{\mathbf{y}}\wedge\check{J}^{\mathbf{y}}=\left(\sum_{j=1}^{n-1}y_jg(E_j'(x_n),E_n(x_n))\right)J_1^{\mathbf{y}}\wedge\cdots\wedge J_n^{\mathbf{y}},$$

where

$$(J_1^{\mathbf{y}}\wedge\cdots\wedge J_n^{\mathbf{y}})(\rho)=\rho^n(\mathbf{e}_1^{\mathbf{y}}\wedge\cdots\wedge\mathbf{e}_n^{\mathbf{y}})(\rho)+O(\rho^{n+2}),$$

thus

$$\rho^{n-1}\|\partial_1\mathbf{r}\wedge\cdots\wedge\partial_n\mathbf{r}\|(\mathbf{x})=\|J_1^{\mathbf{y}}\wedge\cdots\wedge J_{n-1}^{\mathbf{y}}\wedge\hat{J}^{\mathbf{y}}\|(\rho)+\left(\sum_{j=1}^{n-1}y_jg(E_j'(x_n),E_n(x_n))\right)\|J_1^{\mathbf{y}}\wedge\cdots\wedge J_n^{\mathbf{y}}\|(\rho).$$

The orthogonality of the vector fields E_i and $E_n=\gamma'$ gives the equation $g(E_i',\gamma')+g(E_i,\gamma'')=0$, hence

$$\sum_{j=1}^{n-1}y_jg(E_j'(x_n),E_n(x_n))=-g\left(\sum_{j=1}^{n-1}y_jE_j(x_n),\gamma''(x_n)\right).$$

The volume density function $\omega\colon U\to\mathbb{R}$ of the exponential map of M is defined on the domain $U\subset TM$ of the exponential map Exp by the following condition: if $p\in M$, then the pull-back of the Riemannian volume measure by Exp_p is $\omega|_{T_pM\cap U}$ times the Lebesgue measure on T_pM. It is known that

$$\omega(x_1E_1(x_n)+\cdots+x_{n-1}E_{n-1}(x_n))=\|J_1^{\mathbf{x}}\wedge\cdots\wedge J_n^{\mathbf{x}}\|(1)=\rho^{-n}\|J_1^{\mathbf{y}}\wedge\cdots\wedge J_n^{\mathbf{y}}\|(\rho).$$

Denote the left hand side of this equation by $\overline{\omega}(\mathbf{x})$.

Let $\tilde{J}^{\mathbf{x}}$ be the Jacobi field along $\eta^{\mathbf{x}}$ defined by the conditions $\tilde{J}^{\mathbf{x}}(0) = E_n(x_n)$ and $\tilde{J}^{\mathbf{x}}(1) = \mathbf{0}$. $\hat{J}^{\mathbf{x}}$ can be decomposed as

$$\hat{J}^{\mathbf{x}} = \tilde{J}^{\mathbf{x}} - \sum_{i=1}^{n} g(\tilde{J}^{\mathbf{x}\prime}(0), E_i(x_n))J_i^{\mathbf{x}}.$$

Using this decomposition, we obtain

$$\big(J_1^{\mathbf{x}} \wedge \cdots \wedge J_{n-1}^{\mathbf{x}} \wedge \hat{J}^{\mathbf{x}}\big)(1) = -g(\tilde{J}^{\mathbf{x}\prime}(0), E_n(x_n))\big(J_1^{\mathbf{x}} \wedge \cdots \wedge J_n^{\mathbf{x}}\big)(1).$$

If ρ is small, then $g(\tilde{J}^{\mathbf{x}\prime}(0), E_n(x_n)) = -1/\rho + O(1) < 0$, thus the volume of the tube is

$$\mathrm{vol}(\mathcal{T}(\gamma, r)) = \int_0^l \int_{B_r^{n-1}} \left(-g(\tilde{J}^{\mathbf{x}\prime}(0), \gamma'(x_n)) - g\left(\sum_{i=1}^{n-1} x_i E_i(x_n), \gamma''(x_n)\right)\right) \varpi(\mathbf{x})\,\mathrm{d}\mathbf{x}. \tag{3}$$

Differentiating the function $t \mapsto \mathrm{vol}(\mathcal{T}(\gamma|_{[0,t]}, r))$ at $t = x_n$, we obtain that in a manifold having the tube property with function V, equation

$$V(r) = \int_{B_r^{n-1}} \left(-g(\tilde{J}^{\mathbf{x}\prime}(0), \gamma'(x_n)) - g\left(\sum_{i=1}^{n-1} x_i E_i(x_n), \gamma''(x_n)\right)\right) \varpi(\mathbf{x})\,\mathrm{d}x_1 \ldots \mathrm{d}x_{n-1} \tag{4}$$

holds for any unit speed curve $\gamma\colon [0, l] \to M$ and any $x_n \in [0, l]$.

Let $\mathbf{u} \in T_pM$ be an arbitrary unit tangent vector at a point $p \in M$. Define $B_r^{n-1}(\mathbf{u})$ to be the $(n-1)$-ball

$$B_r^{n-1}(\mathbf{u}) = \{\mathbf{w} \in T_pM \mid g(\mathbf{u}, \mathbf{w}) = 0 \text{ and } \|\mathbf{w}\| \le r\}.$$

For $\mathbf{w} \in B_r^{n-1}(\mathbf{u})$, let $\tilde{J}_{\mathbf{u}}^{\mathbf{w}}$ denote the Jacobi field along the geodesic curve $t \mapsto \mathrm{Exp}(t\mathbf{w})$ defined by $\tilde{J}_{\mathbf{u}}^{\mathbf{w}}(0) = \mathbf{u}$ and $\tilde{J}_{\mathbf{u}}^{\mathbf{w}}(1) = \mathbf{0}$. $\tilde{J}_{\mathbf{u}}^{\mathbf{w}}$ is uniquely defined as r is small.

Since for any choice of $\mathbf{u}, \mathbf{v} \in T_pM$ satisfying $\|\mathbf{u}\| = 1$ and $g(\mathbf{u}, \mathbf{v}) = 0$, we can find a unit speed curve $\gamma\colon [0, l] \to M$ and a parameter $x_n \in (0, l)$ such that $\gamma(x_n) = p$, $\gamma'(x_n) = \mathbf{u}$, and $\gamma''(x_n) = \mathbf{v}$, (4) gives

$$V(r) = -\int_{B_r^{n-1}(\mathbf{u})} g(\tilde{J}_{\mathbf{u}}^{\mathbf{w}\prime}(0), \mathbf{u})\omega(\mathbf{w})\,\mathrm{d}\mathbf{w} - \int_{B_r^{n-1}(\mathbf{u})} g(\mathbf{w}, \mathbf{v})\omega(\mathbf{w})\,\mathrm{d}\mathbf{w}. \tag{5}$$

If the manifold M and the radius r are fixed, then the first integral of the right hand side of (5) depends only on the vector $\mathbf{u}$, while the second only on $\mathbf{u}$ and $\mathbf{v}$. Substituting $\mathbf{v} = \mathbf{0}$ into (5), we obtain

$$V(r) = -\int_{B_r^{n-1}(\mathbf{u})} g(\tilde{J}_{\mathbf{u}}^{\mathbf{w}\prime}(0), \mathbf{u})\omega(\mathbf{w})\,\mathrm{d}\mathbf{w}, \tag{6}$$

which implies

$$0 = \int_{B_r^{n-1}(\mathbf{u})} g(\mathbf{w}, \mathbf{v})\omega(\mathbf{w})\,\mathrm{d}\mathbf{w} \tag{7}$$

for any allowed choice of $\mathbf{u}$ and $\mathbf{v}$. Equation (5) characterizing spaces with the tube property is equivalent to the pair of equations (6) and (7). Finding the geometrical meaning of the latter equations leads us to the following theorem.

Theorem 1. *A Riemannian manifold has the tube property if and only if it is a D'Atri space and satisfies the tube property for geodesic curves.*

Proof. As $\gamma'' \equiv 0$ for geodesic curves, equation (6) holds in a Riemannian manifold if and only if the manifold has the tube property for geodesics.

As the density function ω is even for D'Atri spaces, and the linear function $\ell_{\mathbf{v}}\colon T_pM \to \mathbb{R}$, $\ell_{\mathbf{v}}(\mathbf{w}) = g(\mathbf{w}, \mathbf{v})$ is odd, (7) holds in every D'Atri space.

It remains to show that (7) implies that the space is D'Atri. If we have $\int_{B_r^{n-1}(\mathbf{u})} \ell_{\mathbf{v}}(\mathbf{w})\omega(\mathbf{w})\,\mathrm{d}\mathbf{w} = 0$ for all linear functions $\ell_{\mathbf{v}}$, where $\mathbf{u} \perp \mathbf{v}$, then the equation holds also without the orthogonality assumption. Differentiation with respect to r gives that

$$\int_{S_r^{n-2}(\mathbf{u})} \ell_{\mathbf{v}}(\mathbf{w})\omega(\mathbf{w})\,\mathrm{d}\mathbf{w} = 0, \tag{8}$$

where $S_r^{n-2}(\mathbf{u})$ is the boundary sphere of $B_r^{n-1}(\mathbf{u})$, and $\mathrm{d}\mathbf{w}$ stands for integration with respect to the hypersurface measure of the sphere $S_r^{n-2}(\mathbf{u})$. Denote by $B_r^n(p) \subset T_pM$ the ball of radius r about the origin of T_pM, and by $S_r^{n-1}(p)$ its boundary sphere. Equation (8) means that the Funk transform of the restriction of the function $\ell_{\mathbf{v}}\omega$ onto $S_r^{n-1}(p)$ is 0. It is known that a smooth function on a sphere is in the kernel of the Funk transform if and only if it is odd (see Theorem 1.7 in [16, p. 93]). This implies that $\ell_{\mathbf{v}}\omega$ is an odd function on the sphere $S_r^{n-1}(p)$. As this is true for any small r, we conclude that $\ell_{\mathbf{v}}\omega$ is an odd function on the ball $B_r^{n-1}(p)$. As $\ell_{\mathbf{v}}$ is an arbitrary linear function, we have that ω is an even function on the ball $B_r^{n-1}(p)$. This means that the manifold is a D'Atri space. □

Remark. L. Vanhecke and T. J. Willmore conjectured in [6, p. 38] that if equation

$$\mathrm{vol}(\mathcal{T}(\gamma,r)) = \int_0^l \int_{B_r^{n-1}} -g(\check{J}^{\mathbf{x}\prime}(0), \gamma'(x_n))\overline{\omega}(\mathbf{x})\,\mathrm{d}\mathbf{x} \tag{9}$$

holds for an arbitrary unit speed curve γ, then the space is D'Atri. Comparing (9) to (3), we see that (9) implies

$$0 = \int_0^l \int_{B_r^{n-1}} -g\left(\sum_{i=1}^{n-1} x_i E_i(x_n), \gamma''(x_n)\right)\overline{\omega}(\mathbf{x})\,\mathrm{d}\mathbf{x}.$$

Differentiating with respect to l at $l = x_n$ and choosing γ so that $\gamma(x_n) = p$, $\gamma'(x_n) = \mathbf{u}$ and $\gamma''(x_n) = \mathbf{v}$, we obtain that (7) holds in the space. As we have seen, this implies that the space is D'Atri.

The general formula for the volume of a tube about a curve can be simplified in the case of a geodesic curve. First of all, as γ' is parallel for a geodesic, we may choose the orthonormal frame $E_1, \dots, E_n$ to be parallel. Then the Jacobi fields $\check{J}^{\mathbf{x}}$ are equal to zero. Writing $\mathbf{x}$ in the form $\mathbf{x} = (\rho\mathbf{u}, t)$, where $\mathbf{u} \in S_1^{n-2}$ is a unit vector, $\rho \geq 0$ and setting $\mathbf{y} = (\mathbf{u}, t)$, we obtain

$$\mathrm{vol}(\mathcal{T}(\gamma,r)) = \int_{B_r^{n-1}\times[0,l]} \|J_1^{\mathbf{x}}\wedge\cdots\wedge J_{n-1}^{\mathbf{x}}\wedge\hat{J}^{\mathbf{x}}\|(1)\,\mathrm{d}\mathbf{x} = \int_0^l\int_0^r\int_{S_1^{n-2}} \frac{1}{\rho}\|J_1^{\mathbf{y}}\wedge\cdots\wedge J_{n-1}^{\mathbf{y}}\wedge\hat{J}^{\mathbf{y}}\|(\rho)\,\mathrm{d}\mathbf{u}\,\mathrm{d}\rho\,\mathrm{d}t. \tag{10}$$

3. Tube property in harmonic manifolds

The main goal of this section is to prove the following theorem.

Theorem 2. *Every connected harmonic manifold has the tube property.*

Proof. Let (M, g) denote a connected harmonic manifold. As a harmonic manifold is a D'Atri space, it is enough to prove the tube property for geodesic curves of M by Theorem 1. This reduces to showing that the integral on the right hand side of (6) does not depend on the unit tangent vector $\mathbf{u}$.

When the exponential map of M is defined on the closed Euclidean ball $B_r^n(p) \subset T_pM$, then $\mathcal{B}_r(p) = \mathrm{Exp}(B_r^n(p))$ is the geodesic ball of radius r centered at p. A geodesic half-ball is the exponential image of a Euclidean half-ball in a tangent space centered at the origin.

Definition. For a tangent vector $\mathbf{v} \in T_pM \setminus \{\mathbf{0}\}$ and a radius r less than the injectivity radius of M at p, we define the half-ball $\mathcal{H}_r(\mathbf{v})$ by the formula

$$\mathcal{H}_r(\mathbf{v}) = \mathrm{Exp}(\{\mathbf{w} \in T_pM \mid g(\mathbf{v},\mathbf{w}) \geq 0 \text{ and } \|\mathbf{w}\| \leq r\}).$$

Proposition 1. *In a D'Atri space, the volume of a small geodesic half-ball depends only on the radius.*

Proof. The volume preserving geodesic reflection in p maps the half-ball $\mathcal{H}_r(\mathbf{v})$ onto its complementary half-ball $\mathcal{H}_r(-\mathbf{v})$. Consequently, a geodesic half-ball has the same volume as its complementary half-ball. We also know that in a D'Atri space, the volume of a small geodesic ball depends only on the radius of the ball [17]. Hence the volume of a half-ball also depends only on the radius. □

Fix an arbitrary unit tangent vector $\mathbf{u} \in T_pM$ and consider the unit speed geodesic curve γ in M, starting at $\gamma(0) = p$ with initial velocity $\gamma'(0) = \mathbf{u}$. Let

$$N_r(t) = \{q \in M \mid \exists\tau \in [0,t],\ d(q,\gamma(\tau)) \leq r\}$$

be the r-neighborhood of $\gamma([0,t])$, where $t > 0$ is a small number, d is the intrinsic metric of M induced by g.

Lemma 1. *$N_r(t)$ can be decomposed into the non-overlapping union of two geodesic half-balls and a tube as follows*

$$N_r(t) = \mathcal{H}_r(-\gamma'(0)) \cup \mathcal{T}(\gamma|_{[0,t]}, r) \cup \mathcal{H}_r(\gamma'(t)).$$

Proof. The inclusion $\supseteq$ is trivial. If $q \in N_r(t)$, then let $\gamma(\tau)$, $\tau \in [0,t]$ be the closest point of $\gamma([0,t])$ to q, and let η be the shortest geodesic connecting $\gamma(\tau)$ to q. If $\tau = 0$, then by the formula for the first variation of arclength, η has to enclose with γ an obtuse or right angle, therefore $q \in \mathcal{H}_r(-\gamma'(0))$. Similarly, $\tau = t$ gives $q \in \mathcal{H}_r(\gamma'(t))$. Finally, if $0 < \tau < t$, then the variation formula implies that η is orthogonal to γ, consequently $q \in \mathcal{T}(\gamma|_{[0,t]}, r)$. $\square$

Lemma 1 gives that $\mathrm{vol}(N_r(t)) = \mathrm{vol}(\mathcal{H}_r(-\gamma'(0))) + \mathrm{vol}(\mathcal{T}(\gamma|_{[0,t]}, r) + \mathrm{vol}(\mathcal{H}_r(\gamma'(t)))$. This equality, equation (3) and Proposition 1 yield that

$$-\int_{B_r^{n-1}(\mathbf{u})} g(\tilde{J}_{\mathbf{u}}^{\mathbf{w}}{}'(0), \mathbf{u})\omega(\mathbf{w})\,\mathrm{d}\mathbf{w} = \frac{\mathrm{d}}{\mathrm{d}t}\mathrm{vol}(\mathcal{T}(\gamma|_{[0,t]}, r))\Big|_{t=0} = \frac{\mathrm{d}}{\mathrm{d}t}\mathrm{vol}(N_r(t))\Big|_{t=0}. \tag{11}$$

Consider the domain $E_r(t) = \mathcal{B}_r(\gamma(0)) \cup \mathcal{B}_r(\gamma(t))$.

Lemma 2. *We have*

$$N_r(t) \supseteq E_r(t) \supseteq \mathcal{H}_r(-\gamma'(0)) \cup \mathcal{T}(\gamma|_{[0,t]}, r - t/2) \cup \mathcal{H}_r(\gamma'(t)).$$

Proof. The first inclusion is a corollary of the definition of $N_r(t)$. To show the second one, choose a point q in $\mathcal{H}_r(-\gamma'(0)) \cup \mathcal{T}(\gamma|_{[0,t]}, r-t/2) \cup \mathcal{H}_r(\gamma'(t))$. It is clear that if q is in one of the half-balls $\mathcal{H}_r(-\gamma'(0))$ or $\mathcal{H}_r(\gamma'(t))$, then $q \in E_r(t)$. If $q \in \mathcal{T}(\gamma|_{[0,t]}, r-t/2)$, then there is a $\tau \in [0,t]$ such that $d(q, \gamma(\tau)) \leq r - t/2$. If $\tau \leq t/2$, then the triangle inequality gives

$$d(q, \gamma(0)) \leq d(q, \gamma(\tau)) + d(\gamma(\tau), \gamma(0)) \leq (r - t/2) + t/2 = r,$$

thus $q \in \mathcal{B}_r(\gamma(0)) \subseteq E_r(t)$. Similarly, if $\tau \geq t/2$, then $q \in \mathcal{B}_r(\gamma(t)) \subseteq E_r(t)$. $\square$

The above two lemmata give that

$$\begin{aligned}|\mathrm{vol}(N_r(t)) - \mathrm{vol}(E_r(t))| &\leq \mathrm{vol}(\mathcal{T}(\gamma|_{[0,t]}, r)) - \mathrm{vol}(\mathcal{T}(\gamma|_{[0,t]}, r - t/2)) \\ &= -\int_0^t \int_{r-t/2}^r \int_{S_\rho^{n-2}(\gamma'(\tau))} g(\tilde{J}_{\gamma'(\tau)}^{\mathbf{w}}{}'(0), \gamma'(\tau))\omega(\mathbf{w})\,\mathrm{d}\mathbf{w}\,\mathrm{d}\rho\,\mathrm{d}\tau.\end{aligned}$$

By the mean value theorem for integration, there is a point $(\bar{\tau}, \bar{\rho})$ in the rectangle $[0,t] \times [r - t/2, r]$ such that

$$\int_0^t \int_{r-t/2}^r \int_{S_\rho^{n-2}(\gamma'(\tau))} g(\tilde{J}_{\gamma'(\tau)}^{\mathbf{w}}{}'(0), \gamma'(\tau))\omega(\mathbf{w})\,\mathrm{d}\mathbf{w}\,\mathrm{d}\rho\,\mathrm{d}\tau = \frac{t^2}{2}\int_{S_{\bar{\rho}}^{n-2}(\gamma'(\bar{\tau}))} g(\tilde{J}_{\gamma'(\bar{\tau})}^{\mathbf{w}}{}'(0), \gamma'(\bar{\tau}))\omega(\mathbf{w})\,\mathrm{d}\mathbf{w}.$$

Choose $0 < T < r/2$ such that γ is defined on $[0,T]$ and denote by C the maximum of

$$\int_{S_\rho^{n-2}(\gamma'(\tau))} g(\tilde{J}_{\gamma'(\tau)}^{\mathbf{w}}{}'(0), \gamma'(\tau))\omega(\mathbf{w})\,\mathrm{d}\mathbf{w}$$

as (τ, ρ) is running over the rectangle $[0,T] \times [r/2, r]$. Then for any $0 < t < T$, we have

$$|\mathrm{vol}(N_r(t)) - \mathrm{vol}(E_r(t))| \leq \frac{C}{2}t^2.$$

From this, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{vol}(N_r(t))\Big|_{t=0} = \frac{\mathrm{d}}{\mathrm{d}t}\mathrm{vol}(E_r(t))\Big|_{t=0}. \tag{12}$$

In a harmonic manifold, the volume of the union of two small geodesic balls depends only on the radius and the distance between their centers [18]. Hence we have that $\mathrm{vol}(E_r(t))$ depends only on r and t, but not on the fixed geodesic γ, so we can define the function V by $V(r) = \frac{\mathrm{d}}{\mathrm{d}t}\mathrm{vol}(E_r(t))\big|_{t=0}$. Equations (11) and (12) show that a harmonic manifold has the tube property with the function V. $\square$

4. The volume of a tube in a Damek–Ricci space

There are two known families of harmonic manifolds, the two-point homogeneous spaces and the Damek–Ricci spaces. A. Gray and L. Vanhecke computed the volume of tubes in two-point homogeneous spaces [3]. In this section, we compute the volume of tubes in Damek–Ricci spaces.

Let $\mathfrak{n} = \mathfrak{v} \oplus \mathfrak{z}$ be a generalized Heisenberg algebra ($\dim \mathfrak{v} = p, \dim \mathfrak{z} = q$). Recall that $\mathfrak{n}$ is a two-step nilpotent Lie algebra with center $\mathfrak{z} = [\mathfrak{n}, \mathfrak{n}]$, endowed with an inner product $\langle\, ,\rangle$ and a map $J\colon \mathfrak{z} \to \mathrm{End}(\mathfrak{v})$, $Z \mapsto J_Z$ such that

$$\begin{aligned}
\forall V \in \mathfrak{v},\, Z \in \mathfrak{z} &: \langle V, Z\rangle = 0,\\
\forall V_1, V_2 \in \mathfrak{v},\, Z \in \mathfrak{z} &: \langle J_Z V_1, V_2\rangle = \langle [V_1, V_2], Z\rangle,\\
\forall V \in \mathfrak{v},\, Z \in \mathfrak{z} &: J_Z^2(V) = -\langle Z, Z\rangle V.
\end{aligned}$$

Let $\mathfrak{a}$ be a one-dimensional real vector space with a basis element A. Consider the direct sum $\mathfrak{s} = \mathfrak{n} \oplus \mathfrak{a}$ of the linear spaces $\mathfrak{n}$ and $\mathfrak{a}$. We can write an element of $\mathfrak{s}$ as $V + Z + tA$, where $V \in \mathfrak{v}$, $Z \in \mathfrak{z}$, $t \in \mathbb{R}$. Extend the inner product $\langle\, ,\rangle$ and the Lie bracket $[\, ,]$ of $\mathfrak{n}$ onto $\mathfrak{s}$ by

$$\begin{aligned}
\langle V_1 + Z_1 + t_1 A, V_2 + Z_2 + t_2 A\rangle &= \langle V_1 + Z_1, V_2 + Z_2\rangle + t_1 t_2,\\
[V_1 + Z_1 + t_1 A, V_2 + Z_2 + t_2 A] &= [V_1, V_2] + \frac{1}{2} t_1 V_2 - \frac{1}{2} t_2 V_1 + t_1 Z_2 - t_2 Z_1.
\end{aligned}$$

The simply connected Lie group attached to the Lie algebra $\mathfrak{s}$, equipped with the induced left-invariant metric, is called a Damek–Ricci space S. One can show that S is a semi-direct product $N \rtimes \mathbb{R}$, where N is the generalized Heisenberg group attached to $\mathfrak{n}$. Hence we can write a point of S in the form $(\exp_{\mathfrak{n}}(V+Z), t)$, where $\exp_{\mathfrak{n}}$ denotes the Lie exponential map of N. Every element of S will be given in this form below. For example, the unit element $\mathbb{1}$ of S is $(\exp_{\mathfrak{n}}(0), 0)$. The multiplication rule of S is

$$(\exp_{\mathfrak{n}}(V_1 + Z_1), t_1) \cdot (\exp_{\mathfrak{n}}(V_2 + Z_2), t_2) = (\exp_{\mathfrak{n}}(V_1 + e^{\frac{t_1}{2}} V_2 + Z_1 + e^{t_1} Z_2 + \frac{1}{2} e^{\frac{t_1}{2}} [V_1, V_2]), t_1 + t_2).$$

The Riemannian metric is

$$g_{(\exp_{\mathfrak{n}}(V+Z),t)}(V_1 + Z_1 + t_1 A, V_2 + Z_2 + t_2 A) = e^{-t}\langle V_1, V_2\rangle + e^{-2t}\langle Z_1 - \frac{1}{2}[V, V_1], Z_2 - \frac{1}{2}[V, V_2]\rangle + t_1 t_2.$$

The pull-back of the volume measure of S onto $\mathfrak{v} \oplus \mathfrak{z} \oplus \mathfrak{a}$ is $\varrho(V + Z + tA)\, \mathrm{d}V\, \mathrm{d}Z\, \mathrm{d}t$, where $\varrho(V + Z + tA) = e^{-(\frac{p}{2}+q)t}$ is the volume density function, $\mathrm{d}V\, \mathrm{d}Z\, \mathrm{d}t$ is the Lebesgue measure on $\mathfrak{v} \oplus \mathfrak{z} \oplus \mathfrak{a}$.

For more details on Damek–Ricci spaces, see [19] or [20].

As every Damek–Ricci space is harmonic, to compute the volume of a tube of small radius about an arbitrary curve, it is enough to compute that volume for a single geodesic curve by Theorem 2. We will do the computation for the geodesic curve $\gamma\colon \mathbb{R} \to S$, $\gamma(t) = \mathrm{Exp}(tA) = (\exp_{\mathfrak{n}}(0), t)$. The left translation Φ_t by $\gamma(t)$ is an isometry of S for all $t \in \mathbb{R}$. The geodesic γ is an orbit of the one parameter group $\Phi_* = \{\Phi_t : t \in \mathbb{R}\}$ of these isometries. Introduce the following notions.

$$\begin{aligned}
B_r^{n-1} &= \{W \in T_{\mathbb{1}} S : \langle W, A\rangle = 0,\ \|W\| \le r\},\\
\mathcal{B}_r^{n-1} &= \mathrm{Exp}(B_r^{n-1}).
\end{aligned}$$

Let a be small enough to assure that the last coordinate of every point of $\Phi_a(\mathcal{B}_r^{n-1})$ with respect to the semidirect product decomposition $N \rtimes \mathbb{R}$ is negative. We can choose b in a similar way to get that the last coordinate of every point of $\Phi_b(\mathcal{B}_r^{n-1})$ is positive. The tube of radius r about the geodesic segment $\gamma|_{[a,b]}$ is the set $T = \bigcup_{a \le t \le b} \Phi_t(\mathcal{B}_r^{n-1})$. Consider the intersection Σ of hypersurface $(\exp_{\mathfrak{n}}(\mathfrak{n}), 0)$ and the tube T, and let $\sigma \subset \mathfrak{n}$ be the subset defined by $\exp_{\mathfrak{n}}(\sigma) \times \{0\} = \Sigma$. For any $x \in \mathcal{B}_r^{n-1}$, Σ intersects the curve $\{\Phi_t(x) : a \le t \le b\}$ at exactly one point. T is split into two pieces $T = T_- \cup T_+$ by Σ, where the pieces $T_\pm = \{(w, t) \in T : \pm t \ge 0\}$ are defined by the sign of the last coordinate. As

$$\bigcup_{0 \le t \le b-a} \Phi_t(\Sigma) = T_+ \cup \Phi_{b-a}(T_-),$$

and Φ_{b-a} preserves the volume, we have

$$\mathrm{vol}(T) = \mathrm{vol}\left(\bigcup_{0\le t\le b-a} \Phi_t(\Sigma)\right).$$

To compute the volume on the right hand side, define the sets

$$\sigma_t = \{W \in \mathfrak{n} \mid (\exp_{\mathfrak{n}}(W), t) \in \Phi_t(\Sigma)\} = \{e^{\frac{t}{2}}V + e^t Z \mid V \in \mathfrak{v},\, Z \in \mathfrak{z},\, V + Z \in \sigma\}.$$

Then

$$\mathrm{vol}\left(\bigcup_{0\le t\le b-a} \Phi_t(\Sigma)\right) = \int_0^{b-a}\int_{\sigma_t} \varrho(W + tA)\,\mathrm{d}W\,\mathrm{d}t = \int_0^{b-a}\int_{\sigma_t} e^{-(\frac{p}{2}+q)t}\,\mathrm{d}W\,\mathrm{d}t.$$

Computing the inner integral using the linear substitution $\sigma \to \sigma_t$, $V + Z \mapsto e^{\frac{t}{2}}V + e^t Z$, we obtain

$$\int_{\sigma_t} e^{-(\frac{p}{2}+q)t}\,\mathrm{d}W = \int_{\sigma} e^{-(\frac{p}{2}+q)t}e^{(\frac{p}{2}+q)t}\,\mathrm{d}W = \mathrm{vol}(\sigma).$$

Thus,

$$\mathrm{vol}(T) = (b-a)\mathrm{vol}(\sigma). \tag{13}$$

To describe the shape of σ, find the intersection of Σ and the Φ_* orbit of a typical point of $\mathcal{B}_r^{n-1}$. Consider a unit speed geodesic γ_{V+Z} starting from the unit element $\gamma_{V+Z}(0) = \mathbb{1}$ with initial velocity $\gamma'_{V+Z}(0) = V+Z$, perpendicular to γ, where $\|V + Z\| = 1$, $\|Z\| = z$. We have

$$\gamma_{V+Z}(t) = \left(\exp_{\mathfrak{n}}\left(\frac{2\theta(t)}{\chi(t)}V + \frac{2\theta^2(t)}{\chi(t)}J_Z V + \frac{2\theta(t)}{\chi(t)}Z\right), \log\left(\frac{1-\theta^2(t)}{\chi(t)}\right)\right),$$

where $\theta(t) = \tanh\left(\frac{t}{2}\right)$ and $\chi(t) = 1 + z^2\theta^2(t)$ (see Section 4.1.11 in [19]).

The left translation by Φ_t moves the boundary point $\gamma_{V+Z}(r)$ of $\mathcal{B}_r^{n-1}$ to Σ if and only if $t = -\log\left(\frac{1-\theta^2(r)}{\chi(r)}\right)$. Then $\Phi_t(\gamma_{V+Z}(r))$ is the point $(\exp_{\mathfrak{n}}(P_{V+Z}), 0)$, where

$$\begin{aligned} P_{V+Z} &= \sqrt{\frac{\chi(r)}{1-\theta^2(r)}}\frac{2\theta(r)}{\chi(r)}V + \sqrt{\frac{\chi(r)}{1-\theta^2(r)}}\frac{2\theta^2(r)}{\chi(r)}J_Z V + \frac{\chi(r)}{1-\theta^2(r)}\frac{2\theta(r)}{\chi(r)}Z \\ &= \frac{2\theta(r)}{\sqrt{(1-\theta^2(r))\chi(r)}}V + \frac{2\theta^2(r)}{\sqrt{(1-\theta^2(r))\chi(r)}}J_Z V + \frac{2\theta(r)}{1-\theta^2(r)}Z. \end{aligned}$$

The squared norms of the $\mathfrak{v}$ and $\mathfrak{z}$ components of P_{V+Z} are equal to

$$\begin{aligned} \left\|\frac{2\theta(r)}{\sqrt{(1-\theta^2(r))\chi(r)}}V + \frac{2\theta^2(r)}{\sqrt{(1-\theta^2(r))\chi(r)}}J_Z V\right\|^2 &= \left(\frac{4\theta^2(r)}{(1-\theta^2(r))\chi(r)} + \frac{4\theta^4(r)}{(1-\theta^2(r))\chi(r)}z^2\right)\|V\|^2 \\ &= \frac{4\theta^2(r)}{1-\theta^2(r)}\|V\|^2 = 4\sinh^2\left(\frac{r}{2}\right)\|V\|^2 \end{aligned}$$

and

$$\left\|\frac{2\theta(r)}{1-\theta^2(r)}Z\right\|^2 = 4\sinh^2\left(\frac{r}{2}\right)\cosh^2\left(\frac{r}{2}\right)\|Z\|^2.$$

As $V + Z$ runs over the unit sphere of $\mathfrak{n}$, P_{V+Z} runs over the ellipsoid in $\mathfrak{n}$ defined by the equation

$$\frac{\|V\|^2}{4\sinh^2\left(\frac{r}{2}\right)} + \frac{\|Z\|^2}{4\sinh^2\left(\frac{r}{2}\right)\cosh^2\left(\frac{r}{2}\right)} = 1.$$

This ellipsoid has p semi-principal axes of length $2\sinh\left(\frac{r}{2}\right)$ and q semi-principal axes of length $2\sinh\left(\frac{r}{2}\right)\cosh\left(\frac{r}{2}\right)$. Since for any fixed Z of length less than or equal to 1, the $\mathfrak{v}$ component of P_{V+Z} is obtained from V by a linear similarity transformation, the unit sphere of $\mathfrak{n}$ is mapped *onto* this ellipsoid. Therefore σ is the body bounded by this ellipsoid, and its volume is

$$\mathrm{vol}(\sigma) = \omega_{p+q}2^{p+q}\sinh^{p+q}\left(\frac{r}{2}\right)\cosh^q\left(\frac{r}{2}\right),$$

where ω_m denotes the volume of the m-dimensional Euclidean unit ball for $m \in \mathbb{N}$. Substituting this volume into (13), we obtain the following

Theorem 3. *The volume of a solid tube of radius r about a curve of length l in a Damek–Ricci space is*

$$\omega_{p+q} 2^{p+q} \sinh^{p+q}\left(\frac{r}{2}\right) \cosh^{q}\left(\frac{r}{2}\right) l.$$

The $p+q$-dimensional volume of the tubular surface of radius r about the curve is

$$\omega_{p+q} 2^{p+q-1} \left((p+q) \sinh^{p+q-1}\left(\frac{r}{2}\right) \cosh^{q+1}\left(\frac{r}{2}\right) + q \sinh^{p+q+1}\left(\frac{r}{2}\right) \cosh^{q-1}\left(\frac{r}{2}\right) \right) l.$$

The second part of the theorem follows from the first part by differentiating with respect to the radius r.

The theorem generalizes earlier result of A. Gray and L. Vanhecke on the volume of tubes in rank one non-compact symmetric spaces [3], as $\mathbb{C}H^n$, $\mathbb{H}H^n$, and $\mathbb{O}H^2$ are Damek–Ricci spaces with parameters $(p,q) = (2n-2, 1)$, $(4n-4, 3)$, and $(8,7)$ respectively.

5. Manifolds with the tube property are 2-stein

The main result of this section is the following.

Theorem 4. *A manifold having the tube property is a 2-stein space.*

We recall that a Riemannian manifold is said to be 2-stein if the manifold is Einstein and there exists a constant λ such that

$$\operatorname{tr}(R_{\mathbf{u}}^2) = \lambda \|\mathbf{u}\|^4$$

for every tangent vector $\mathbf{u}$, where for $\mathbf{u} \in T_pM$, $R_{\mathbf{u}} \colon T_pM \to T_pM$, $R_{\mathbf{u}}(\mathbf{x}) = R(\mathbf{x}, \mathbf{u})\mathbf{u}$ is the Jacobi operator.

For the proof, we need an elementary lemma.

Lemma 3. *Let $P \in \mathbb{R}[x,y]$ be a polynomial of degree $k \geq 1$ in two variables, $P = P_0 + \dots + P_k$ be its decomposition into homogeneous components. If the function $\theta \mapsto P(\cos\theta, \sin\theta)$ is constant, then $P_k(1, \mathbb{i}) = 0$, where $\mathbb{i} \in \mathbb{C}$ is the imaginary unit.*

Proof. Since the polynomial $P - P(1,0)$ vanishes on the circle $x^2 + y^2 - 1 = 0$, the irreducible polynomial $x^2 + y^2 - 1$ divides $P - P(1,0)$, i.e., there is a polynomial $G \in \mathbb{R}[x,y]$ such that $P(x,y) - P(1,0) = (x^2 + y^2 - 1)G(x,y)$. Considering the highest degree homogeneous component of both sides, we obtain $P_k(x,y) = (x^2 + y^2)G_{k-2}(x,y)$, where G_{k-2} is the degree $k-2$ homogeneous part of G. Substituting $(x,y) = (1, \mathbb{i})$ into the last equation gives $P_k(1, \mathbb{i}) = 0$. □

Proof of Theorem 4. Denote by τ, ρ, and R the scalar curvature, the Ricci tensor, and the Riemannian curvature tensor respectively. A. Gray and L. Vanhecke [3] computed the initial terms of the Taylor series of the volume of tubes about a curve using Fermi coordinates. Recall that the Fermi coordinate system on the tube $\mathcal{T}(\gamma, r)$ about the injective unit speed curve $\gamma \colon [0,l] \to M$ is the inverse of the parameterization $\tilde{\mathbf{r}} \colon [0,l] \times B_r^{n-1} \to M$, $\tilde{\mathbf{r}}(x_1, x_2, \dots, x_n) = \mathbf{r}(x_2, \dots, x_n, x_1)$, where $\mathbf{r}$ is the parameterization defined in (1). They obtained the formula

$$\operatorname{vol}(\mathcal{T}(\gamma, r)) = \omega_{n-1} r^{n-1} \int_0^l \left(1 + Ar^2 + Br^4 + O(r^6)\right)(\gamma(t))\, \mathrm{d}t$$

with coefficients

$$A = -\frac{1}{6(n+1)}(\tau + \rho_{11}),$$

$$B = \frac{1}{360(n+1)(n+3)}\Big(-18\Delta\tau + 5\tau^2 + 8\|\rho\|^2 - 3\|R\|^2 + 33\nabla_{11}^2\tau - 9\Delta\rho_{11} + 10\tau\rho_{11} + 2\sum_{i=2}^{n} \rho_{1i}^2 +$$
$$+ 14 \sum_{i,j=2}^{n} \rho_{ij} R_{1i1j} - 6 \sum_{i,j,k=2}^{n} R_{1ijk}^2 - 21\nabla_{11}^2\rho_{11} - 3\rho_{11}^2 - 10 \sum_{i,j=2}^{n} R_{1i1j}^2 + 60W\tau - 60\nabla_1\rho_{1W} - 30\nabla_W\rho_{11}\Big),$$

where W is a vector field such that $W(\gamma(t)) = \gamma''(t)$, and the tensor coordinates are taken with respect to the Fermi coordinate system.

If the manifold has the tube property, then A and B are constant along γ, and their values do not depend on the curve γ. For an arbitrary point $p \in M$, choose an orthonormal basis $\mathbf{u}_1, \dots, \mathbf{u}_n$ in T_pM. Computing the coefficient A associated to the geodesic curve $t \mapsto \operatorname{Exp}(t\mathbf{u}_i)$ at $t = 0$, we obtain

$$A = -\frac{1}{6(n+1)}(\tau(p) + \rho(\mathbf{u}_i, \mathbf{u}_i)) \quad \text{for } i = 1, \dots, n.$$

Summation for i yields that $\tau = -6nA$, from which $\rho(\mathbf{u}_i, \mathbf{u}_i) = -6A$ for all i, hence the manifold is Einstein.

In the case of a geodesic curve in an Einstein manifold, B can be simplified to

$$B = \frac{1}{360(n+1)(n+3)}\left(5\tau^2 + 8\|\rho\|^2 - 3\|R\|^2 + 10\tau\rho_{11} + 14\rho_{11}^2 - 6\sum_{i,j,k=2}^{n} R_{1ijk}^2 - 3\rho_{11}^2 - 10\sum_{i,j=2}^{n} R_{1i1j}^2\right).$$

From this, we can conclude that in a manifold having tube property, the sum

$$-3\|R\|^2 - 6\sum_{i,j,k=2}^{n} R_{1ijk}^2 - 10\sum_{i,j=2}^{n} R_{1i1j}^2$$

is a constant function along the curve γ and the value C of this constant does not depend on the curve.

Observe that for any point $p \in M$, and for any orthonormal basis $\mathbf{u}_1, \dots, \mathbf{u}_n$ in T_pM, there is a unit speed curve $\gamma\colon [0,l] \to M$ starting at $\gamma(0) = p$ and a Fermi coordinate system around γ such that the basis vector fields induced by the coordinate system are equal to $\mathbf{u}_1, \dots, \mathbf{u}_n$ at p. This means that for any orthonormal basis $\mathbf{u}_1, \dots, \mathbf{u}_n$ of T_pM, the coordinates $R_{ijkl} = R(\mathbf{u}_i, \mathbf{u}_j, \mathbf{u}_k, \mathbf{u}_l)$ of the curvature tensor should satisfy the identity

$$C = -3\|R\|^2 - 6\sum_{i,j,k=2}^{n} R_{1ijk}^2 - 10\sum_{i,j=2}^{n} R_{1i1j}^2 = -3\|R\|^2 - 6\sum_{i,j,k=1}^{n} R_{1ijk}^2 + 2\sum_{i,j=1}^{n} R_{1i1j}^2.$$

In particular, transposing the role of $\mathbf{u}_1$ and $\mathbf{u}_a$, we obtain

$$C = -3\|R\|^2 - 6\sum_{i,j,k=1}^{n} R_{aijk}^2 + 2\sum_{i,j=1}^{n} R_{aiaj}^2 \quad \text{for } 1 \le a \le n.$$

Introduce the tensor fields P and Q by their components

$$P_{ab} = \sum_{i,j,k=1}^{n} R_{aijk}R_{bijk},$$

$$Q_{abcd} = \sum_{i,j=1}^{n} R_{aibj}R_{cidj}.$$

We have the identities

$$P_{ab} = P_{ba}, \qquad Q_{abcd} = Q_{badc} = Q_{cdab},$$

$$\sum_{b=1}^{n} Q_{abab} = P_{aa}, \quad \sum_{b=1}^{n} Q_{abba} = \frac{1}{2}P_{aa}, \quad \sum_{a=1}^{n} P_{aa} = \|R\|^2.$$

In an Einstein manifold (with $\rho = Kg$), we have

$$\sum_{b=1}^{n} Q_{aabb} = K^2.$$

With this notation, we can write C as

$$C = -3\|R\|^2 - 6P_{aa} + 2Q_{aaaa}. \tag{14}$$

Equivalently,

$$C = -3\|R\|^2 - 6P(\mathbf{u}, \mathbf{u}) + 2Q(\mathbf{u}, \mathbf{u}, \mathbf{u}, \mathbf{u}) \tag{15}$$

for any unit tangent vector $\mathbf{u}$. Summing (14) over a gives

$$nC = -3n\|R\|^2 - 6\|R\|^2 + 2\sum_{a=1}^{n} Q_{aaaa},$$

so we have

$$\sum_{a=1}^{n} Q_{aaaa} = \frac{n}{2}C + \frac{3n+6}{2}\|R\|^2. \tag{16}$$

Assume that the tensor components are taken with respect to the orthonormal basis $\mathbf{u}_1, \ldots, \mathbf{u}_n$ at p. Apply equation (15) for the unit tangent vector $\mathbf{u}(\theta) = \cos\theta\mathbf{u}_a + \sin\theta\mathbf{u}_b$, $(a \neq b)$. Then we get that the function $f(\theta) = -3P(\mathbf{u}(\theta), \mathbf{u}(\theta)) + Q(\mathbf{u}(\theta), \mathbf{u}(\theta), \mathbf{u}(\theta), \mathbf{u}(\theta))$ is constant. The function f is a polynomial of degree 4 of $\cos\theta$ and $\sin\theta$, and its degree 4 homogeneous term is $Q(\mathbf{u}(\theta), \mathbf{u}(\theta), \mathbf{u}(\theta), \mathbf{u}(\theta))$. Applying Lemma 3, we obtain

$$Q(\mathbf{u}_a + \mathbb{i}\mathbf{u}_b, \mathbf{u}_a + \mathbb{i}\mathbf{u}_b, \mathbf{u}_a + \mathbb{i}\mathbf{u}_b, \mathbf{u}_a + \mathbb{i}\mathbf{u}_b) = 0.$$

The real part of this equation is

$$Q_{aaaa} + Q_{bbbb} = 2(Q_{aabb} + Q_{abab} + Q_{abba}). \tag{17}$$

Sum (17) for all b not equal to a, and add $6Q_{aaaa}$ to both sides. Then we obtain

$$(n+4)Q_{aaaa} + \sum_{b=1}^{n} Q_{bbbb} = 2\left(K^2 + P_{aa} + \frac{1}{2}P_{aa}\right).$$

Using (14) and (16), we get

$$(n+4)Q_{aaaa} + \frac{n}{2}C + \frac{3n+6}{2}\|R\|^2 = 2K^2 - \frac{3}{2}\|R\|^2 + Q_{aaaa} - \frac{C}{2},$$

that is

$$(n+3)Q_{aaaa} + \frac{n+1}{2}C + \frac{3n+9}{2}\|R\|^2 = 2K^2,$$

which gives

$$Q_{aaaa} = -\frac{3}{2}\|R\|^2 + \frac{2}{n+3}K^2 - \frac{n+1}{2(n+3)}C. \tag{18}$$

If we sum (18) for a, we obtain

$$\sum_{a=1}^{n} Q_{aaaa} = -\frac{3n}{2}\|R\|^2 + \frac{2n}{n+3}K^2 - \frac{n^2+n}{2(n+3)}C. \tag{19}$$

Equations (16) and (19) show that

$$-\frac{3n}{2}\|R\|^2 + \frac{2n}{n+3}K^2 - \frac{n^2+n}{2(n+3)}C = \frac{n}{2}C + \frac{3n+6}{2}\|R\|^2,$$

which implies that $\|R\|^2$ is constant on the manifold. By equation (18), we can conclude that $Q_{aaaa} = \operatorname{tr}(R^2_{\mathbf{u}_a})$ is constant, which means that the manifold is 2-stein. We remark that equation (14) implies also that P_{aa} is constant. □

6. Tube property in symmetric spaces

In this section, we prove that a symmetric space has the tube property if and only if it is harmonic. Using that Jacobi fields can be computed in a symmetric space, first we transform formula (10) to a more explicit form, see equation (21) below.

6.1. **Volume of tubes about geodesics in a symmetric space.** Consider a unit speed geodesic curve $\gamma\colon [0,l]\to M$ in a symmetric space M, and fix a parallel orthonormal frame $E_1,\dots,E_n$ along it such that $E_n=\gamma'$. The volume of a tube about γ is given by (10). This formula uses Jacobi fields along geodesic curves starting from a point of γ orthogonally to γ'.

Fix a point $p=\gamma(t)$ and a unit vector $\mathbf{u}\in T_pM$ such that $\mathbf{u}\perp\gamma'(t)$. Let η be the geodesic starting at $\eta(0)=p$ with initial velocity $\eta'(0)=\mathbf{u}$. Denote by $\mathbf{e}_1,\dots,\mathbf{e}_n$ the parallel orthonormal frame along η extending the orthonormal basis $E_1(t),\dots,E_n(t)$ of $T_{\eta(0)}M$. If X is a vector field along η, then we shall denote by $[X]$ the column vector of the coordinate functions of X with respect to the frame $\mathbf{e}_1,\dots,\mathbf{e}_n$. In a symmetric space, the matrix of the Jacobi operator $R_{\eta'}$ with respect to the frame $\mathbf{e}_1,\dots,\mathbf{e}_n$ is constant. Its value at 0 is the matrix $[R_{\mathbf{u}}]$ of the Jacobi operator $R_{\mathbf{u}}=R_{\eta'(0)}$. The coordinate vector of a Jacobi field J along η satisfies the Jacobi differential equation

$$[J]''(t)+[R_{\mathbf{u}}][J](t)=0.$$

The general solution of this equation is

$$[J](t)=\cos\left(\sqrt{[R_{\mathbf{u}}]}t\right)\mathbf{a}+\frac{\sin\left(\sqrt{[R_{\mathbf{u}}]}t\right)}{\sqrt{[R_{\mathbf{u}}]}}\mathbf{b},\qquad \mathbf{a},\mathbf{b}\in\mathbb{R}^n,$$

where trigonometric function symbols abbreviate their power series, i.e.,

$$\cos\left(\sqrt{[R_{\mathbf{u}}]}t\right)=\sum_{k=0}^{\infty}(-1)^k\frac{[R_{\mathbf{u}}]^kt^{2k}}{(2k)!},\qquad \frac{\sin\left(\sqrt{[R_{\mathbf{u}}]}t\right)}{\sqrt{[R_{\mathbf{u}}]}}=\sum_{k=0}^{\infty}(-1)^k\frac{[R_{\mathbf{u}}]^kt^{2k+1}}{(2k+1)!}$$

which make sense without clarifying what the square root of $R_{\mathbf{u}}$ and the inverse of the square root are. Consider the Jacobi fields $J_1^{\mathbf{y}},\dots,J_{n-1}^{\mathbf{y}},\hat{J}^{\mathbf{y}}$ along the geodesic $\eta^{\mathbf{y}}$ and their coordinates with respect to the parallel frame $\mathbf{e}_1^{\mathbf{y}},\dots,\mathbf{e}_n^{\mathbf{y}}$ as described in Section 2. We have $J_i^{\mathbf{y}}(0)=\mathbf{0}$, $J_i^{\mathbf{y}\prime}(0)=\mathbf{e}_i^{\mathbf{y}}(0)$, hence

$$[J_i^{\mathbf{y}}](t)=\frac{\sin\left(\sqrt{[R_{\mathbf{u}}]}t\right)}{\sqrt{[R_{\mathbf{u}}]}}[\mathbf{e}_i^{\mathbf{y}}](0)\quad\text{ for } i=1,\dots,n-1,$$

where $\mathbf{u}=\eta^{\mathbf{y}\prime}(0)$.

As $\hat{J}^{\mathbf{y}}(0)=\mathbf{e}_n^{\mathbf{y}}(0)$ and $\hat{J}^{\mathbf{y}\prime}(0)=\mathbf{0}$,

$$[\hat{J}^{\mathbf{y}}](t)=\cos\left(\sqrt{[R_{\mathbf{u}}]}t\right)[\mathbf{e}_n^{\mathbf{y}}](0)=\sqrt{[R_{\mathbf{u}}]}\cot\left(\sqrt{[R_{\mathbf{u}}]}t\right)\frac{\sin\left(\sqrt{[R_{\mathbf{u}}]}t\right)}{\sqrt{[R_{\mathbf{u}}]}}[\mathbf{e}_n^{\mathbf{y}}](0).$$

Thus,

$$\|J_1^{\mathbf{y}}\wedge\dots\wedge J_{n-1}^{\mathbf{y}}\wedge\hat{J}^{\mathbf{y}}\|(\rho)=\det\left(\frac{\sin\left(\sqrt{R_{\mathbf{u}}}\rho\right)}{\sqrt{R_{\mathbf{u}}}}\right)\cdot\left\langle\sqrt{R_{\mathbf{u}}}\cot\left(\sqrt{R_{\mathbf{u}}}\rho\right)\mathbf{e}_n^{\mathbf{y}}(0),\mathbf{e}_n^{\mathbf{y}}(0)\right\rangle. \tag{20}$$

If the eigenvalues of $R_{\mathbf{u}}$ are $\lambda_1,\dots,\lambda_n$, then

$$\det\left(\frac{\sin(\sqrt{R_{\mathbf{u}}}\rho)}{\sqrt{R_{\mathbf{u}}}\rho}\right)=\prod_{i=1}^{n}\frac{\sin(\sqrt{\lambda_i}\rho)}{\sqrt{\lambda_i}\rho}.$$

Our goal is to express this determinant with the help of the power sums $S_k=\lambda_1^k+\dots+\lambda_n^k=\operatorname{tr}(R_{\mathbf{u}}^k)$.

Let b_k be the coefficient of x^{2k} in the Maclaurin series of the even analytic function $x\cot x$. Then

$$x\cot x=\sum_{k=0}^{\infty}b_kx^{2k}\quad\text{ for } |x|<\pi.$$

It is known that $b_0=1$ and $b_k=(-4)^kB_{2k}/(2k)!<0$ for $k>0$, where B_m denotes the mth Bernoulli number. With this notation, if $|x|<\pi$, then

$$x\left(\log\left(\frac{\sin x}{x}\right)\right)'=x\cot x-1=\sum_{k=1}^{\infty}b_kx^{2k},$$

which gives

$$\log\left(\frac{\sin x}{x}\right)=\sum_{k=1}^{\infty}\frac{b_k}{2k}x^{2k}.$$

Thus, if $\rho\geq 0$ is sufficiently small, namely if $|\sqrt{\lambda_i}\rho|<\pi$ for all eigenvalues λ_i, then

$$\log\left(\prod_{i=1}^{n}\frac{\sin(\sqrt{\lambda_i}\rho)}{\sqrt{\lambda_i}\rho}\right)=\sum_{k=1}^{\infty}\frac{b_k}{2k}\sum_{i=1}^{n}\lambda_i^k\rho^{2k}=\sum_{k=1}^{\infty}\frac{b_k}{2k}S_k\rho^{2k},$$

hence

$$\det\left(\frac{\sin(\sqrt{R_{\mathbf{u}}}\rho)}{\sqrt{R_{\mathbf{u}}}\rho}\right)=\prod_{i=1}^{n}\frac{\sin(\sqrt{\lambda_i}\rho)}{\sqrt{\lambda_i}\rho}=\sum_{m=0}^{\infty}\frac{1}{m!}\left(\sum_{k=1}^{\infty}\frac{b_k}{2k}S_k\rho^{2k}\right)^m=\sum_{m=0}^{\infty}\frac{1}{m!}\left(\sum_{k=1}^{\infty}\frac{b_k}{2k}\mathrm{tr}(R_{\mathbf{u}}^k)\rho^{2k}\right)^m.$$

Combining this equation with (20) and (10), we obtain that for small values of r,
(21)

$$\mathrm{vol}(\mathcal{T}(\gamma,r))=\int_0^l\int_0^r\int_{S_1^{n-2}(\gamma'(t))}\rho^{n-2}\left(\sum_{m=0}^{\infty}\frac{1}{m!}\left(\sum_{k=1}^{\infty}\frac{b_k}{2k}\mathrm{tr}(R_{\mathbf{u}}^k)\rho^{2k}\right)^m\right)\sum_{k=0}^{\infty}b_k\langle R_{\mathbf{u}}^k(\gamma'(t)),\gamma'(t)\rangle\rho^{2k}\,\mathrm{d}\mathbf{u}\,\mathrm{d}\rho\,\mathrm{d}t.$$

6.2. **Symmetric spaces having the tube property.** Now we prove the following theorem.

Theorem 5. *Every symmetric space having the tube property is either Euclidean or has rank one.*

We remark that a symmetric space is harmonic if and only if it has rank one or it is Euclidean (see, e.g., [21]).

The proof is based on the following observation and two lemmas. By equation (21), in a symmetric space having the tube property, the integral

$$\int_{S_1^{n-2}(\mathbf{e})}\rho^{n-2}\left(\sum_{m=0}^{\infty}\frac{1}{m!}\left(\sum_{k=1}^{\infty}\frac{b_k}{2k}\mathrm{tr}(R_{\mathbf{u}}^k)\rho^{2k}\right)^m\right)\sum_{k=0}^{\infty}b_k\langle R_{\mathbf{u}}^k\mathbf{e},\mathbf{e}\rangle\rho^{2k}\,\mathrm{d}\mathbf{u}$$

does not depend on the unit tangent vector $\mathbf{e}$. Taking the coefficients of the Taylor series expansion with respect to ρ, we get that for all positive integers k, the integral

$$\int_{S_1^{n-2}(\mathbf{e})}\frac{b_k}{2k}\mathrm{tr}(R_{\mathbf{u}}^k)+\Big\{\sum_{\substack{0\leq l<k,\\ 1\leq l_1,\dots,l_m<k\\ l+\sum_{i=1}^m l_i=k}}\frac{b_l}{m!}\langle R_{\mathbf{u}}^l\mathbf{e},\mathbf{e}\rangle\prod_{i=1}^{m}\frac{b_{l_i}}{2l_i}\mathrm{tr}(R_{\mathbf{u}}^{l_i})\Big\}+b_k\langle R_{\mathbf{u}}^k\mathbf{e},\mathbf{e}\rangle\,\mathrm{d}\mathbf{u}\tag{22}$$

is also independent of the unit tangent vector $\mathbf{e}$.

We prove Theorem 5 by contradiction. Assume that there is a nonflat symmetric space $M=G/K$ of rank $r>1$ having the tube property, where G is the identity component of the isometry group of M, and K is the stabilizer of a point $o\in M$. If I_o is the geodesic reflection in the point o, then K is an open subgroup of the fixed point set of the involutive automorphism $\sigma\in\mathrm{Aut}(G)$, $\sigma(h)=I_o\circ h\circ I_o$. Let $\mathfrak{g}$ and $\mathfrak{k}$ be the Lie algebras of G and K. The derivative map $T\sigma$ of σ gives an involutive automorphism $s=T\sigma|_{\mathfrak{g}}$ of the Lie algebra $\mathfrak{g}$. Setting $\mathfrak{p}=\{\mathbf{v}\in\mathfrak{g}\mid s(\mathbf{v})=-\mathbf{v}\}$, we have $\mathfrak{g}=\mathfrak{k}\oplus\mathfrak{p}$. There is a natural isomorphism $T_oM\cong\mathfrak{p}$, which induces an inner product $\langle\,,\rangle$ on $\mathfrak{p}$. The pair $(\mathfrak{g},s)$ is an orthogonal symmetric Lie algebra.

By Theorem 4, M is 2-stein, and consequently, it is (locally) irreducible (see [13]). In particular, M is a symmetric space of either compact or non-compact type, and $\langle\,,\rangle$ is a constant multiple of the restriction of the Killing form of $\mathfrak{g}$ onto $\mathfrak{p}$. This constant multiple of the Killing form extends $\langle\,,\rangle$ to a non-degenerate invariant symmetric bilinear function on $\mathfrak{g}$, which will also be denoted by $\langle\,,\rangle$.

The rank of M is the dimension of a maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{p}$. Choose an orthonormal basis $\mathbf{a}_1,\mathbf{a}_2,\dots,\mathbf{a}_r$ of $\mathfrak{a}$, and complete it to an orthonormal basis of $\mathfrak{p}$ with $\mathbf{a}_{r+1},\dots,\mathbf{a}_n$. Let

$$\begin{aligned}\mathbf{b}_1&=\cos\theta\,\mathbf{a}_1+\sin\theta\,\mathbf{a}_2,\\ \mathbf{b}_2&=-\sin\theta\,\mathbf{a}_1+\cos\theta\,\mathbf{a}_2,\\ \mathbf{b}_i&=\mathbf{a}_i\text{ if }i\geq 3.\end{aligned}$$

When we want to emphasize that $\mathbf{b}_1$ and $\mathbf{b}_2$ depend on θ, we shall denote them by $\mathbf{b}_1(\theta)$ and $\mathbf{b}_2(\theta)$.

Our plan to prove Theorem 5 is to evaluate the integral (22) for $\mathbf{e} = \mathbf{e}(\theta) = \mathbf{b}_1(\theta)$ and exploit its independence of θ.

With the identification $T_oM \cong \mathfrak{p}$, the Jacobi operator $R_\mathbf{u}$ can be expressed as

$$R_\mathbf{u} = -\mathrm{ad}^2(\mathbf{u})|_\mathfrak{p}.$$

Decompose $\mathbf{u} \perp \mathbf{b}_1$ as $\mathbf{u} = \sum_{i=2}^n u_i \mathbf{b}_i$ and denote by $\bar{\mathbf{u}} = \sum_{i>r} u_i \mathbf{b}_i$ the projection of $\mathbf{u}$ onto the orthogonal complement of $\mathfrak{a}$. If $L\colon V \to V$ is a linear endomorphism of the linear space V and $W \leq V$ is an L-invariant linear subspace of V, then denote by $\mathrm{tr}_W(L)$ the trace of the restriction $L|_W$ of L onto W. We have

$$\mathrm{tr}(R_\mathbf{u}^k) = (-1)^k \mathrm{tr}_\mathfrak{p}(\mathrm{ad}^{2k}(\mathbf{u})) = (-1)^k \mathrm{tr}_\mathfrak{p}(\mathrm{ad}^{2k}(u_2\mathbf{b}_2(\theta) + \sum_{i=3}^n u_i \mathbf{b}_i)).$$

If $k \geq 1$, then $\langle R_\mathbf{u}^k \mathbf{b}_1, \mathbf{b}_1\rangle$ seems to be a degree $2k+2$ trigonometric polynomial of θ, however, the following identities show that its degree is at most $2k$.

$$\begin{aligned}
\langle R_\mathbf{u}^k \mathbf{b}_1, \mathbf{b}_1\rangle &= (-1)^k \left\langle \mathrm{ad}^{2k}\left(\sum_{i=2}^n u_i\mathbf{b}_i\right)(\mathbf{b}_1), \mathbf{b}_1 \right\rangle \\
&= (-1)^{k+1} \left\langle \mathrm{ad}^{2k-2}\left(\sum_{i=2}^n u_i\mathbf{b}_i\right) \circ \mathrm{ad}(\bar{\mathbf{u}})(\mathbf{b}_1), \mathrm{ad}(\bar{\mathbf{u}})(\mathbf{b}_1) \right\rangle \\
&= (-1)^k \left\langle \mathrm{ad}(\mathbf{b}_1(\theta)) \circ \mathrm{ad}^{2k-2}\left(u_2\mathbf{b}_2(\theta) + \sum_{i=3}^n u_i\mathbf{b}_i\right) \circ \mathrm{ad}(\mathbf{b}_1(\theta))(\bar{\mathbf{u}}), \bar{\mathbf{u}} \right\rangle.
\end{aligned}$$

The last expression is clearly a polynomial of degree $\leq 2k$ of $\cos\theta$ and $\sin\theta$. This means that the integral in (22) is also a polynomial of degree $\leq 2k$ of $\cos\theta$ and $\sin\theta$. To apply Lemma 3, we compute the degree $2k$ homogeneous components of these polynomials of $\cos\theta$ and $\sin\theta$. The above equations yield

$$\begin{aligned}
\mathrm{tr}(R_\mathbf{u}^k) &= (-1)^k u_2^{2k} \mathrm{tr}_\mathfrak{p}(\mathrm{ad}^{2k}(\mathbf{b}_2(\theta))) + \dots, \\
\langle R_\mathbf{u}^k \mathbf{b}_1, \mathbf{b}_1\rangle &= (-1)^k \left\langle \mathrm{ad}^2(\mathbf{b}_1(\theta)) \circ \mathrm{ad}^{2k-2}(u_2\mathbf{b}_2(\theta))(\bar{\mathbf{u}}), \bar{\mathbf{u}} \right\rangle + \dots,
\end{aligned}$$

where $\dots$ stands for a polynomial of degree less than $2k$ of $\cos\theta$ and $\sin\theta$.

We will need the following formula for the integral of monomials over the unit sphere (see [22]).

Proposition 2. *Let $P(x_1, \dots, x_n) = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ be a monomial in n variables, and let $\beta_j = \frac{1}{2}(\alpha_j + 1)$. Then*

$$\int_{S_1^{n-1}} P(\mathbf{u})\,\mathrm{d}\mathbf{u} = \begin{cases} 0 & \text{if some } \alpha_j \text{ is odd,} \\ \frac{2\Gamma(\beta_1)\cdots\Gamma(\beta_n)}{\Gamma(\beta_1+\cdots+\beta_n)} & \text{if all } \alpha_j \text{ are even.} \end{cases}$$

Lemma 4. *If $\mathbf{a}_1, \mathbf{a}_2 \in \mathfrak{a}$ are the vectors introduced above, then $\mathrm{tr}_{\mathbb{C}\otimes\mathfrak{p}}\left(\mathrm{ad}^{2k}(\mathbf{a}_1 + \mathbb{i}\mathbf{a}_2)\right) = 0$ for all positive integers k.*

Proof. We prove the lemma by induction on k. To show the base case $k = 1$, consider the integral in (22) for $k = 1$ and $\mathbf{e} = \mathbf{b}_1$. We obtain that the value of the integral modulo a trigonometric polynomial of θ of degree less than 2 is equal to

$$\begin{aligned}
&\int_{S_1^{n-2}(\mathbf{b}_1)} -\frac{b_1}{2} u_2^2 \mathrm{tr}_\mathfrak{p}(\mathrm{ad}^2(\mathbf{b}_2(\theta))) - b_1 \left\langle \mathrm{ad}^2(\mathbf{b}_1(\theta))(\bar{\mathbf{u}}), \bar{\mathbf{u}} \right\rangle \mathrm{d}\mathbf{u} \\
&\qquad = C\left(-\frac{b_1}{2}\mathrm{tr}_\mathfrak{p}(\mathrm{ad}^2(-\sin\theta\,\mathbf{a}_1 + \cos\theta\,\mathbf{a}_2)) - b_1 \mathrm{tr}_\mathfrak{p}(\mathrm{ad}^2(\cos\theta\,\mathbf{a}_1 + \sin\theta\,\mathbf{a}_2))\right),
\end{aligned}$$

where $C > 0$ is the integral of the monomial x_1^2 over the unit sphere S_1^{n-2}. The value of the integral in (22) for $\mathbf{e} = \mathbf{b}_1$ does not depend on the choice of θ, so we can apply Lemma 3 to it. This yields

$$0 = -\mathrm{tr}_{\mathbb{C}\otimes\mathfrak{p}}(\mathrm{ad}^2(-\mathbb{i}\mathbf{a}_1 + \mathbf{a}_2)) - 2\mathrm{tr}_{\mathbb{C}\otimes\mathfrak{p}}(\mathrm{ad}^2(\mathbf{a}_1 + \mathbb{i}\mathbf{a}_2)) = -\mathrm{tr}_{\mathbb{C}\otimes\mathfrak{p}}(\mathrm{ad}^2(\mathbf{a}_1 + \mathbb{i}\mathbf{a}_2)),$$

which settles the base case.

For the induction step, assume that $k \geq 2$ and the statement is true for positive integers less than k. Evaluate the integral in (22) for k and $\mathbf{e} = \mathbf{b}_1$. Modulo trigonometric polynomials of θ of degree less than $2k$, the integral equals

$$
\begin{aligned}
(-1)^k \int_{S_1^{n-2}(\mathbf{b}_1)} & \frac{b_k}{2k} u_2^{2k} \mathrm{tr}_{\mathfrak{p}}(\mathrm{ad}^{2k}(\mathbf{b}_2(\theta))) + \sum_{\substack{1 \leq l_1,\dots,l_m < k \\ \sum_{i=1}^m l_i = k}} \frac{1}{m!} \prod_{i=1}^m \frac{b_{l_i}}{2l_i} u_2^{2l_i} \mathrm{tr}_{\mathfrak{p}}(\mathrm{ad}^{2l_i}(\mathbf{b}_2(\theta))) + \\
&+ \sum_{\substack{0<l<k, \\ 1 \leq l_1,\dots,l_m < k \\ l+\sum_{i=1}^m l_i = k}} \frac{b_l}{m!} \left\langle \mathrm{ad}^2(\mathbf{b}_1(\theta)) \circ \mathrm{ad}^{2l-2}(u_2 \mathbf{b}_2(\theta))\,(\bar{\mathbf{u}})\,, \bar{\mathbf{u}} \right\rangle \prod_{i=1}^m \frac{b_{l_i}}{2l_i} u_2^{2l_i} \mathrm{tr}_{\mathfrak{p}}(\mathrm{ad}^{2l_i}(\mathbf{b}_2(\theta))) + \\
&+ b_k \left\langle \mathrm{ad}^2(\mathbf{b}_1(\theta)) \circ \mathrm{ad}^{2k-2}(u_2 \mathbf{b}_2(\theta))(\bar{\mathbf{u}}), \bar{\mathbf{u}} \right\rangle \, \mathrm{d}\mathbf{u} \\
= (-1)^k \Bigg(& C_1 \frac{b_k}{2k} \mathrm{tr}_{\mathfrak{p}}(\mathrm{ad}^{2k}(\mathbf{b}_2(\theta))) + C_1 \sum_{\substack{1 \leq l_1,\dots,l_m < k \\ \sum_{i=1}^m l_i = k}} \frac{1}{m!} \prod_{i=1}^m \frac{b_{l_i}}{2l_i} \mathrm{tr}_{\mathfrak{p}}(\mathrm{ad}^{2l_i}(\mathbf{b}_2(\theta))) + \\
&+ C_2 \sum_{\substack{0<l<k, \\ 1 \leq l_1,\dots,l_m < k \\ l+\sum_{i=1}^m l_i = k}} \frac{b_l}{m!} \mathrm{tr}_{\mathfrak{p}}(\mathrm{ad}^2(\mathbf{b}_1(\theta)) \circ \mathrm{ad}^{2l-2}(\mathbf{b}_2(\theta))) \prod_{i=1}^m \frac{b_{l_i}}{2l_i} \mathrm{tr}_{\mathfrak{p}}(\mathrm{ad}^{2l_i}(\mathbf{b}_2(\theta))) + \\
&+ C_2 b_k \mathrm{tr}_{\mathfrak{p}}(\mathrm{ad}^2(\mathbf{b}_1(\theta)) \circ \mathrm{ad}^{2k-2}(\mathbf{b}_2(\theta))) \Bigg),
\end{aligned}
$$

where C_1 and C_2 are the integrals of the monomials x_1^{2k} and $x_1^{2k-2}x_2^2$ over the unit sphere S_1^{n-2} respectively. Just as in the base case, the value of the integral in (22) for $\mathbf{e} = \mathbf{b}_1(\theta)$ does not depend on θ, so we can apply Lemma 3 to it. This gives that

$$
\begin{aligned}
0 = C_1 & \frac{b_k}{2k} \mathrm{tr}_{\mathbb{C}\otimes\mathfrak{p}}(\mathrm{ad}^{2k}(-\mathrm{i}\mathbf{a}_1 + \mathbf{a}_2)) + C_1 \sum_{\substack{1 \leq l_1,\dots,l_m < k \\ \sum_{i=1}^m l_i = k}} \frac{1}{m!} \prod_{i=1}^m \frac{b_{l_i}}{2l_i} \mathrm{tr}_{\mathbb{C}\otimes\mathfrak{p}}(\mathrm{ad}^{2l_i}(-\mathrm{i}\mathbf{a}_1 + \mathbf{a}_2)) + \\
&+ C_2 \sum_{\substack{0<l<k, \\ 1 \leq l_1,\dots,l_m < k \\ l+\sum_{i=1}^m l_i = k}} \frac{b_l}{m!} \mathrm{tr}_{\mathbb{C}\otimes\mathfrak{p}}(\mathrm{ad}^2(\mathbf{a}_1 + \mathrm{i}\mathbf{a}_2) \circ \mathrm{ad}^{2l-2}(-\mathrm{i}\mathbf{a}_1 + \mathbf{a}_2)) \prod_{i=1}^m \frac{b_{l_i}}{2l_i} \mathrm{tr}_{\mathbb{C}\otimes\mathfrak{p}}(\mathrm{ad}^{2l_i}(-\mathrm{i}\mathbf{a}_1 + \mathbf{a}_2)) + \\
&+ C_2 b_k \mathrm{tr}_{\mathbb{C}\otimes\mathfrak{p}}(\mathrm{ad}^2(\mathbf{a}_1 + \mathrm{i}\mathbf{a}_2) \circ \mathrm{ad}^{2k-2}(-\mathrm{i}\mathbf{a}_1 + \mathbf{a}_2)).
\end{aligned}
$$

By the induction hypothesis,

$$
\mathrm{tr}_{\mathbb{C}\otimes\mathfrak{p}}(\mathrm{ad}^{2l_i}(-\mathrm{i}\mathbf{a}_1 + \mathbf{a}_2)) = (-1)^{l_i} \mathrm{tr}_{\mathbb{C}\otimes\mathfrak{p}}(\mathrm{ad}^{2l_i}(\mathbf{a}_1 + \mathrm{i}\mathbf{a}_2)) = 0
$$

if $l_i < k$. For this reason,

$$
\begin{aligned}
0 &= C_1 \frac{b_k}{2k} \mathrm{tr}_{\mathbb{C}\otimes\mathfrak{p}}(\mathrm{ad}^{2k}(-\mathrm{i}\mathbf{a}_1 + \mathbf{a}_2)) + C_2 b_k \mathrm{tr}_{\mathbb{C}\otimes\mathfrak{p}}(\mathrm{ad}^2(\mathbf{a}_1 + \mathrm{i}\mathbf{a}_2) \circ \mathrm{ad}^{2k-2}(-\mathrm{i}\mathbf{a}_1 + \mathbf{a}_2)) \\
&= (-1)^k \left(\frac{C_1}{2k} - C_2 \right) b_k \mathrm{tr}_{\mathbb{C}\otimes\mathfrak{p}}(\mathrm{ad}^{2k}(\mathbf{a}_1 + \mathrm{i}\mathbf{a}_2)).
\end{aligned}
$$

The last equation completes the proof of the lemma as b_k is negative, and by Proposition 2, we have

$$
\frac{C_1}{C_2} = \frac{\Gamma(k+\frac{1}{2})\Gamma(\frac{1}{2})}{\Gamma(k-\frac{1}{2})\Gamma(\frac{3}{2})} = 2k - 1 \neq 2k. \qquad \square
$$

Lemma 5. *Let $(\mathfrak{g}, s)$ be an orthogonal symmetric Lie algebra of compact or non-compact type, $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the Cartan decomposition of $\mathfrak{g}$. Then if $\mathbf{a}_1, \mathbf{a}_2 \in \mathfrak{p}$ are two commuting elements such that $\mathrm{tr}_{\mathbb{C}\otimes\mathfrak{p}}(\mathrm{ad}^{2k}(\mathbf{a}_1 + \mathrm{i}\mathbf{a}_2)) = 0$ for all positive integers k, then $\mathbf{a}_1 = \mathbf{a}_2 = \mathbf{0}$.*

Proof. By duality, orthogonal symmetric Lie algebras of compact and non-compact type occur in dual pairs. If $(\mathfrak{g}, s)$ is an orthogonal symmetric Lie algebra, then its dual is $(\mathfrak{g}^*, s^*)$, where $\mathfrak{g}^* = \mathfrak{k} \oplus \mathrm{i}\mathfrak{p} < \mathbb{C} \otimes \mathfrak{g}$ and

$s^*|_{\mathfrak{k}} = \mathrm{id}|_{\mathfrak{k}}$, $s^*|_{\mathbb{i}\mathfrak{p}} = -\mathrm{id}|_{\mathbb{i}\mathfrak{p}}$. If $\mathbf{a}_1, \mathbf{a}_2$ are two commuting elements in $\mathfrak{p}$, then $\mathbb{i}\mathbf{a}_1, \mathbb{i}\mathbf{a}_2$ are two commuting elements in $\mathbb{i}\mathfrak{p}$, and $\mathrm{tr}_{\mathbb{C}\otimes\mathfrak{p}}(\mathrm{ad}^{2k}(\mathbf{a}_1 + \mathbb{i}\mathbf{a}_2)) = (-1)^k \mathrm{tr}_{\mathbb{C}\otimes\mathbb{i}\mathfrak{p}}(\mathrm{ad}^{2k}((\mathbb{i}\mathbf{a}_1) + \mathbb{i}(\mathbb{i}\mathbf{a}_2)))$. Thus, the lemma is true for $(\mathfrak{g}, s)$ if and only if it is true for its dual. Consequently, we may assume that $(\mathfrak{g}, s)$ is of non-compact type.

Let $\mathfrak{a} < \mathfrak{p}$ be a maximal abelian Lie subalgebra in $\mathfrak{p}$ containing the vectors $\mathbf{a}_1$ and $\mathbf{a}_2$. We construct the Iwasawa decomposition of $\mathfrak{g}$ as it is described in [23, ch. VI, §3]. First we extend $\mathfrak{a}$ to a maximal abelian subalgebra $\mathfrak{h}$ of $\mathfrak{g}$. The Lie algebra $\mathfrak{h}$ decomposes as $\mathfrak{h} = \mathfrak{b} \oplus \mathfrak{a}$, where $\mathfrak{b} = \mathfrak{h} \cap \mathfrak{k}$. The complexification $\mathfrak{h}_{\mathbb{C}} = \mathbb{C} \otimes \mathfrak{h}$ of $\mathfrak{h}$ is a Cartan subalgebra of the complex semisimple Lie algebra $\mathfrak{g}_{\mathbb{C}} = \mathbb{C} \otimes \mathfrak{g}$. Denote by Δ the root system of $\mathfrak{g}_{\mathbb{C}}$ with respect to $\mathfrak{h}_{\mathbb{C}}$, and let

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{h}_{\mathbb{C}} \oplus \bigoplus_{\lambda \in \Delta} \mathfrak{g}_\lambda$$

be the root space decomposition of $\mathfrak{g}_{\mathbb{C}}$.

The roots are real valued on $\mathfrak{h}_{\mathbb{R}} = \mathbb{i}\mathfrak{b} \oplus \mathfrak{a}$, so Δ can be embedded into the dual space of the real linear space $\mathfrak{h}_{\mathbb{R}}$. Select compatible orderings of the dual spaces of $\mathfrak{a}$ and $\mathfrak{h}_{\mathbb{R}}$, this way we get an ordering of Δ. Let Δ^+ denote the set of positive roots. Denote by $\Delta_{\mathfrak{p}}$ the set of roots that do not vanish identically on $\mathfrak{a}$, and put $\Delta_{\mathfrak{p}}^+ = \Delta_{\mathfrak{p}} \cap \Delta^+$. Then the complex nilpotent Lie algebra $\mathfrak{n}_{\mathbb{C}} = \bigoplus_{\lambda \in \Delta_{\mathfrak{p}}^+} \mathfrak{g}_\lambda$ is the complexification of the real nilpotent Lie algebra $\mathfrak{n} = \mathfrak{n}_{\mathbb{C}} \cap \mathfrak{g}$, and $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$. This is the Iwasawa decomposition.

If $\mathbf{a} \in \mathbb{C} \otimes \mathfrak{a}$, then $\mathbb{C} \otimes \mathfrak{k}$ is an $\mathrm{ad}^2(\mathbf{a})$-invariant subspace, and both $\mathbb{C} \otimes \mathfrak{p}$ and $\mathbb{C} \otimes (\mathfrak{a} \oplus \mathfrak{n})$ are $\mathrm{ad}^2(\mathbf{a})$-invariant complementary subspaces to it. Thus, we have

$$\mathrm{tr}_{\mathbb{C}\otimes\mathfrak{p}}(\mathrm{ad}^{2k}(\mathbf{a})) = \mathrm{tr}_{\mathbb{C}\otimes(\mathfrak{a}\oplus\mathfrak{n})}(\mathrm{ad}^{2k}(\mathbf{a})).$$

The subspaces $\mathbb{C} \otimes \mathfrak{a}$ and $\mathfrak{n}_{\mathbb{C}}$ are even $\mathrm{ad}(\mathbf{a})$-invariant. The eigenvalues of the restriction of $\mathrm{ad}(\mathbf{a})$ onto $\mathfrak{n}_{\mathbb{C}}$ are the numbers $\lambda(\mathbf{a})$ for $\lambda \in \Delta_{\mathfrak{p}}^+$, the restriction of $\mathrm{ad}(\mathbf{a})$ onto $\mathbb{C} \otimes \mathfrak{a}$ is zero. From this, we obtain

$$\mathrm{tr}_{\mathbb{C}\otimes(\mathfrak{a}\oplus\mathfrak{n})}(\mathrm{ad}^{2k}(\mathbf{a})) = \sum_{\lambda \in \Delta_{\mathfrak{p}}^+} \lambda^{2k}(\mathbf{a}) = \frac{1}{2} \sum_{\lambda \in \Delta_{\mathfrak{p}}} \lambda^{2k}(\mathbf{a}) = \frac{1}{2} \sum_{\lambda \in \Delta} \lambda^{2k}(\mathbf{a}).$$

Applying this formula for $\mathbf{a} = \mathbf{a}_1 + \mathbb{i}\mathbf{a}_2$, we obtain that $\sum_{\lambda\in\Delta} \lambda^{2k}(\mathbf{a}_1 + \mathbb{i}\mathbf{a}_2) = 0$ for all positive integers k. This implies that $\lambda(\mathbf{a}_1 + \mathbb{i}\mathbf{a}_2) = 0$ for all roots $\lambda \in \Delta$. As $\lambda(\mathbf{a}_1)$ and $\lambda(\mathbf{a}_2)$ are real numbers, we conclude that $\lambda(\mathbf{a}_1) = \lambda(\mathbf{a}_2) = 0$ for all roots λ. Since Δ spans the dual space of the Cartan subalgebra $\mathfrak{h}_{\mathbb{C}}$, this implies $\mathbf{a}_1 = \mathbf{a}_2 = \mathbf{0}$. □

By Lemma 4, we can apply Lemma 5 to the linearly independent basis vectors $\mathbf{a}_1$ and $\mathbf{a}_2$ of $\mathfrak{a}$. However, Lemma 5 implies $\mathbf{a}_1 = \mathbf{a}_2 = \mathbf{0}$, a contradiction.

Since every locally symmetric space is locally isometric to a symmetric space, the following corollary is straightforward.

Corollary 1. *A locally symmetric space has the tube property if and only if it is flat or has rank one.*

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